

MIRZAKHANI'S RECURSION FORMULA IS EQUIVALENT TO THE WITTEN-KONTSEVICH THEOREM

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

ABSTRACT. In this paper, we give a proof of Mirzakhani's recursion formula of Weil-Petersson volumes of moduli spaces of curves using the Witten-Kontsevich theorem. We also describe properties of intersection numbers involving higher degree κ classes.

1. INTRODUCTION

Following the notation of Mulase and Safnuk [20], let $\mathcal{M}_{g,n}(\mathbf{L})$ denote the moduli space of bordered Riemann surfaces with n geodesic boundary components of specified lengths $\mathbf{L} = (L_1, \dots, L_n)$ and let $\text{Vol}_{g,n}(\mathbf{L})$ denote its Weil-Petersson volume $\text{Vol}(\mathcal{M}_{g,n}(\mathbf{L}))$. Using her remarkable generalization of the McShane identity, Mirzakhani [18] proved a beautiful recursion formula for these Weil-Petersson volumes

$$\begin{aligned} \text{Vol}_{g,n}(\mathbf{L}) &= \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ \underline{n}=I \amalg J}} \int_0^{L_1} \int_0^\infty \int_0^\infty xyH(t, x+y) \\ &\quad \times \text{Vol}_{g_1,n_1}(x, \mathbf{L}_I) \text{Vol}_{g_2,n_2}(y, \mathbf{L}_J) dx dy dt \\ &+ \frac{1}{2L_1} \int_0^{L_1} \int_0^\infty \int_0^\infty xyH(t, x+y) \text{Vol}_{g-1,n+1}(x, y, L_2, \dots, L_n) dx dy dt \\ &\quad + \frac{1}{2L_1} \sum_{j=2}^n \int_0^{L_1} \int_0^\infty x(H(x, L_1 + L_j) + H(x, L_1 - L_j)) \\ &\quad \times \text{Vol}_{g,n-1}(x, L_2, \dots, \hat{L}_j, \dots, L_n) dx dt, \end{aligned}$$

where the kernel function

$$H(x, y) = \frac{1}{1 + e^{(x+y)/2}} + \frac{1}{1 + e^{(x-y)/2}}.$$

Using symplectic reduction, Mirzakhani [19] showed the following relation

$$\frac{\text{Vol}_{g,n}(2\pi\mathbf{L})}{(2\pi^2)^{3g+n-3}} = \frac{1}{(3g+n-3)!} \int_{\mathcal{M}_{g,n}} (\kappa_1 + \sum_{i=1}^n L_i^2 \psi_i)^{3g+n-3}$$

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$$= \sum_{\substack{d_0+\dots+d_n \\ =3g+n-3}} \prod_{i=0}^n \frac{1}{d_i!} \langle \kappa_1^{d_0} \prod \tau_{d_i} \rangle_{g,n} \prod_{i=1}^{\infty} L_i^{2d_i}.$$

Combining with her recursion formula of Weil-Petersson volumes, Mirzakhani [19] found a new proof of the celebrated Witten-Kontsevich theorem.

By taking derivatives with respect to $\mathbf{L} = (L_1, \dots, L_n)$ in Mirzakhani's recursion, Mulase and Safnuk [20] obtained the following enlightening recursion formula of intersection numbers which is equivalent to Mirzakhani's recursion.

$$\begin{aligned} & (2d_1 + 1)!! \langle \prod_{j=1}^n \tau_{d_j} \kappa_1^a \rangle_g \\ &= \sum_{j=2}^n \sum_{b=0}^a \frac{a!}{(a-b)!} \frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!} \beta_b \langle \kappa_1^{a-b} \tau_{b+d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\ &+ \frac{1}{2} \sum_{b=0}^a \sum_{r+s=b+d_1-2} \frac{a!}{(a-b)!} (2r+1)!! (2s+1)!! \beta_b \langle \kappa_1^{a-b} \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\ &+ \frac{1}{2} \sum_{b=0}^a \sum_{\substack{c+c'=a-b \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=b+d_1-2} \frac{a!}{c!c'!} (2r+1)!! (2s+1)!! \beta_b \\ &\quad \times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}, \end{aligned}$$

where

$$\beta_b = (2^{2b+1} - 4) \frac{\zeta(2b)}{(2\pi^2)^b} = (-1)^{b-1} 2^b (2^{2b} - 2) \frac{B_{2b}}{(2b)!}.$$

Safnuk [22] gave a proof of the above differential form of Mirzakhani's recursion formula using localization techniques, but he also used the Mirzakhani-McShane formula. The relationship between Mirzakhani's recursion and matrix integrals has been studied by Eynard-Orantin [5] and Eynard [6].

Indeed, when $a = 0$, Mulase-Safnuk differential form of Mirzakhani's recursion is just the Witten-Kontsevich theorem [23, 13] in the form of DVV recursion relation [3]. There are several other new proofs of Witten-Kontsevich theorem [2, 10, 12, 21] besides Mirzakhani's proof [19].

More discussions about Weil-Petersson volumes from the point of view of intersection numbers can be found in the papers [4, 11, 17, 25].

In Section 2, we show that Mirzakhani's recursion formula is essentially equivalent to the Witten-Kontsevich theorem via a formula from [9] expressing κ classes in terms of ψ classes. In Section 3, we present certain results of intersection numbers involving higher degree κ classes.

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2. PROOF OF MIRZAKHANI'S RECURSION FORMULA

We first give three lemmas. The following lemma can be found in [20].

Lemma 2.1. *The constants β_b in Mirzakhani's recursion satisfy the following*

$$\sum_{k=0}^{\infty} \beta_k x^k = \frac{\sqrt{2x}}{\sin \sqrt{2x}}.$$

And its inverse

$$\left(\sum_{k=0}^{\infty} \beta_k x^k\right)^{-1} = \frac{\sin \sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{(2k+1)!} x^k$$

Proof. Since

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n} = \frac{x e^{x/2} + e^{-x/2}}{2 e^{x/2} - e^{-x/2}} = \frac{x}{2i} \cot \frac{x}{2i},$$

we have

$$\sum_{k=0}^{\infty} \beta_k x^k = \sqrt{2x} (\cot \sqrt{\frac{x}{2}} - \cot \sqrt{2x}) = \frac{\sqrt{2x}}{\sin \sqrt{2x}}.$$

□

The following elementary result is crucial to our proof.

Lemma 2.2. *Let $F(m, n)$ and $G(m, n)$ be two functions defined on $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Let α_k and β_k be real numbers that satisfy*

$$\sum_{k=0}^{\infty} \alpha_k x^k = \left(\sum_{k=0}^{\infty} \beta_k x^k\right)^{-1}.$$

Then the following two identities are equivalent.

$$\begin{aligned} G(m, n) &= \sum_{k=0}^m \alpha_k F(m-k, n+k), \quad \forall (m, n) \in \mathbb{N} \times \mathbb{N} \\ F(m, n) &= \sum_{k=0}^m \beta_k G(m-k, n+k), \quad \forall (m, n) \in \mathbb{N} \times \mathbb{N} \end{aligned}$$

Proof. Assume the first identity holds, then we have

$$\begin{aligned} \sum_{i=0}^m \beta_i G(m-i, n+i) &= \sum_{i=0}^m \beta_i \sum_{j=0}^{m-i} \alpha_j F(m-i-j, n+i+j) \\ &= \sum_{k=0}^m \sum_{i+j=k} (\beta_i \alpha_j) F(m-k, n+k) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^m \delta_{k0} F(m-k, n+k) \\
&= F(m, n).
\end{aligned}$$

So we proved the second identity. The proof of the other direction is the same. \square

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [9], and has been developed by Arbarello-Cornalba [1], Faber [7] and Kaufmann-Manin-Zagier [9] into a nice combinatorial formalism.

Lemma 2.3. [9] *For $m > 0$,*

$$\left\langle \prod_{j=1}^n \tau_{d_j} \kappa_1^m \right\rangle_g = \sum_{k=1}^m \frac{(-1)^{m-k}}{k!} \sum_{\substack{m_1+\dots+m_k=m \\ m_i > 0}} \binom{m}{m_1, \dots, m_k} \left\langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{m_j+1} \right\rangle_g.$$

Proof. (sketch) Let $\pi_{n+p,n} : \overline{\mathcal{M}}_{g,n+p} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1} \dots \psi_{n+p}^{a_p+1})$ by $R(a_1, \dots, a_p)$, then we have the formula from [1]

$$R(a_1, \dots, a_p) = \sum_{\sigma \in \mathbb{S}_p} \prod_{\substack{\text{each cycle } c \\ \text{of } \sigma}} \kappa_{\sum_{j \in c} a_j},$$

where we write any permutation σ in the symmetric group \mathbb{S}_p as a product of disjoint cycles.

A formal combinatorial argument [9] leads to the following inversion equation

$$\kappa_{a_1} \dots \kappa_{a_p} = \sum_{k=1}^p \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1, \dots, p\} = S_1 \amalg \dots \amalg S_k \\ S_k \neq \emptyset}} R\left(\sum_{j \in S_1} a_j, \dots, \sum_{j \in S_k} a_j\right),$$

from which the result follows easily. \square

Proposition 2.4.

$$\begin{aligned}
&\sum_{b=0}^a (-1)^b \binom{a}{b} \frac{(2(d_1+b)+1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^n \tau_{d_i} \kappa_1^{a-b} \rangle_g \\
&= \sum_{j=2}^n \frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
&+ \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!! (2s+1)!! \langle \kappa_1^a \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1}
\end{aligned}$$

$$+ \frac{1}{2} \sum_{\substack{c+c'=a \\ I \amalg J = \{2, \dots, n\}}} \binom{a}{c} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.$$

Proof. Let LHS and RHS denote the left and right hand side of the equation respectively. By Lemma 2.3 and the Witten-Kontsevich theorem, we have

$$\begin{aligned} & (2d_1 + 1)!! \langle \prod_{j=1}^n \tau_{d_j} \kappa_1^a \rangle_g \\ &= (2d_1 + 1)!! \sum_{k=0}^a \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1 + \dots + m_k = a \\ m_i > 0}} \binom{a}{m_1, \dots, m_k} \langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{m_j+1} \rangle_g \\ &= \sum_{k=0}^a \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1 + \dots + m_k = a \\ m_i > 0}} \binom{a}{m_1, \dots, m_k} \\ &\quad \times \left(\sum_{j=2}^n \frac{(2(d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \prod_{i=1}^k \tau_{m_i+1} \rangle_g \right. \\ &\quad + \sum_{j=1}^k \frac{(2(d_1 + m_j) + 1)!!}{(2m_j + 1)!!} \langle \tau_{d_1+m_j} \prod_{i=2}^n \tau_{d_i} \prod_{i \neq j} \tau_{m_i+1} \rangle_g \\ &\quad + \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \prod_{i=2}^n \tau_{d_i} \prod_{i=1}^k \tau_{m_i+1} \rangle_{g-1} \\ &\quad + \frac{1}{2} \sum_{\substack{I \amalg J = \{2, \dots, n\} \\ I' \amalg J' = \{1, \dots, k\}}} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \\ &\quad \times \langle \tau_r \prod_{i \in I} \tau_{d_i} \prod_{i \in I'} \tau_{m_i+1} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \prod_{i \in J'} \tau_{m_i+1} \rangle_{g-g'} \Big) \\ &= \sum_{j=2}^n \frac{(2d_1 + 2d_j - 1)!!}{(2d_j - 1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\ &\quad + \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^a \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\ &+ \frac{1}{2} \sum_{\substack{c+c'=a \\ I \amalg J = \{2, \dots, n\}}} \binom{a}{c} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \\ &\quad + \sum_{k=0}^a \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1 + \dots + m_k = a \\ m_i > 0}} \binom{a}{m_1, \dots, m_k} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{j=1}^k \frac{(2(d_1 + m_j) + 1)!!}{(2m_j + 1)!!} \langle \tau_{d_1+m_j} \prod_{i=2}^n \tau_{d_i} \prod_{i \neq j} \tau_{m_i+1} \rangle_g \\
& = RHS + \sum_{k \geq 0} \frac{(-1)^{a-k-1}}{(k+1)!} \sum_{b=1}^a \sum_{\substack{m_1+\dots+m_k=a-b \\ m_i > 0}} \binom{a}{b} \binom{a-b}{m_1, \dots, m_k} \\
& \quad \times (k+1) \frac{(2(d_1+b) + 1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^n \tau_{d_i} \prod_{i=1}^k \tau_{m_i+1} \rangle_g \\
& = RHS - \sum_{b=1}^a (-1)^b \binom{a}{b} \frac{(2(d_1+b) + 1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^n \tau_{d_i} \kappa_1^{a-b} \rangle_g \\
& \quad = RHS - LHS + (2d_1 + 1)!! \langle \prod_{j=1}^n \tau_{d_j} \kappa_1^a \rangle_g.
\end{aligned}$$

So we have proved $RHS = LHS$. \square

Proposition 2.4 is also implicitly contained in the arguments of Mulase and Safnuk [20].

Theorem 2.5.

$$\begin{aligned}
& \frac{(2d_1 + 1)!!}{a!} \langle \prod_{j=1}^n \tau_{d_j} \kappa_1^a \rangle_g \\
& = \sum_{b=0}^a \sum_{j=2}^n \frac{(2(b + d_1 + d_j) - 1)!!}{(a-b)!(2d_j - 1)!!} \beta_b \langle \kappa_1^{a-b} \tau_{b+d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
& \quad + \frac{1}{2} \sum_{b=0}^a \sum_{r+s=b+d_1-2} \frac{(2r+1)!!(2s+1)!!}{(a-b)!} \beta_b \langle \kappa_1^{a-b} \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
& \quad + \frac{1}{2} \sum_{b=0}^a \sum_{\substack{c+c'=a-b \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=b+d_1-2} \frac{(2r+1)!!(2s+1)!!}{c!c'!} \beta_b \\
& \quad \times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},
\end{aligned}$$

where the constants β_k are given by

$$\left(\sum_{k=0}^{\infty} \beta_k x^k \right)^{-1} = \frac{\sin \sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)!!} x^k.$$

Proof. Denote the LHS by $F(a, d_1)$. Let

$$G(a, d_1) = \sum_{j=2}^n \frac{(2(d_1 + d_j) - 1)!!}{a!(2d_j - 1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{a!} \langle \kappa_1^a \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
& + \frac{1}{2} \sum_{\substack{c+c'=a \\ I \amalg J = \{2, \dots, n\}}} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{c!c'!} \times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},
\end{aligned}$$

Note that Proposition 2.4 is just

$$\sum_{b=0}^a \frac{(-1)^b}{b!(2b+1)!!} F(a-b, d_1+b) = G(a, d_1).$$

By Lemmas 2.1 and 2.2, we have

$$F(a, d_1) = \sum_{b=0}^a \beta_b G(a-b, d_1+b) = RHS.$$

So we conclude the proof. \square

3. HIGHER WEIL-PETERSSON VOLUMES

Mirzakhani's formula provides a recursive way of computing the following Weil-Petersson volumes of moduli spaces of curves

$$WP(g) := \int_{\mathcal{M}_{g,n}} \kappa_1^{3g-3+n}.$$

Mirzakhani's formula resorts to intersection numbers of mixed ψ and κ classes.

A natural question is whether there exist an explicit formula expressing $WP(g)$ in terms of those $WP(g')$ with $g' < g$. Recall the following beautiful formula due to Itzykson-Zuber [8].

Proposition 3.1. (Itzykson-Zuber) *Let $g \geq 0$. Then*

$$\phi_{g+1} = \frac{25g^2 - 1}{24} \phi_g + \frac{1}{2} \sum_{m=1}^g \phi_{g+1-m} \phi_m,$$

where $\phi_0 = -1, \phi_1 = \frac{1}{24}$ and

$$\phi_g = \frac{(5g-5)(5g-3)}{2^g(3g-3)!} \langle \tau_2^{3g-3} \rangle_g, \quad g \geq 2.$$

By projection formula, we have

$$\langle \tau_2^{3g-3} \rangle_g = \langle \kappa_1^{3g-3} \rangle_g + \dots,$$

where \dots denote terms involving higher degree kappa classes. Also note that $\langle \kappa_1^{3g-3} \rangle_g$ is conjecturally [15] the largest term in the right hand side.

To our disappointment, so far, all recursion formulae for $WP(g)$ stemming from the Witten-Kontsevich theorem involve either ψ class or higher degree κ classes inevitably.

Mirzakhani, Mulase and Safnuk's arguments use Wolpert's formula [24]

$$\kappa_1 = \frac{1}{2\pi^2} \omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. We have no similar formulae for higher degree κ classes. So a priori κ_1 may be rather special in the intersection theory. However, as we will see, this is not the case.

First we fix notations as in [9]. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m(1), m(2), \dots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i .

Let $\mathbf{m}, \mathbf{t}, \mathbf{a}_1, \dots, \mathbf{a}_n \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a}_i$, $\mathbf{m} \geq \mathbf{t}$ and $\mathbf{s} := (s_1, s_2, \dots)$ be a family of independent formal variables.

$$|\mathbf{m}| := \sum_{i \geq 1} i m(i), \quad \|\mathbf{m}\| := \sum_{i \geq 1} m(i), \quad \mathbf{s}^{\mathbf{m}} := \prod_{i \geq 1} s_i^{m(i)}, \quad \mathbf{m}! := \prod_{i \geq 1} m(i)!,$$

$$\binom{\mathbf{m}}{\mathbf{t}} := \prod_{i \geq 1} \binom{m(i)}{t(i)}, \quad \binom{\mathbf{m}}{\mathbf{a}_1, \dots, \mathbf{a}_n} := \prod_{i \geq 1} \binom{m(i)}{a_1(i), \dots, a_n(i)}.$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i \geq 1} \kappa_i^{b(i)}.$$

We are interested in the following intersection numbers

$$\langle \kappa(\mathbf{b}) \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\mathcal{M}_{g,n}} \kappa(\mathbf{b}) \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

When $d_1 = \cdots = d_n = 0$, these intersection numbers are called higher Weil-Petersson volumes of moduli spaces of curves. The details of the following discussions are contained in [16].

The following lemma is a direct generalization of Lemma 2.2.

Lemma 3.2. *Let $F(\mathbf{L}, n)$ and $G(\mathbf{L}, n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Let $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ be real numbers depending only on $\mathbf{L} \in N^\infty$ that satisfy $\alpha_{\mathbf{0}} \beta_{\mathbf{0}} = 1$ and*

$$\sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \alpha_{\mathbf{L}} \beta_{\mathbf{L}'} = 0, \quad \mathbf{b} \neq \mathbf{0}.$$

Then the following two identities are equivalent.

$$\begin{aligned} G(\mathbf{b}, n) &= \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \alpha_{\mathbf{L}} F(\mathbf{L}', n + |\mathbf{L}|), \quad \forall (\mathbf{b}, n) \in N^\infty \times \mathbb{N} \\ F(\mathbf{b}, n) &= \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \beta_{\mathbf{L}} G(\mathbf{L}', n + |\mathbf{L}|), \quad \forall (\mathbf{b}, n) \in N^\infty \times \mathbb{N} \end{aligned}$$

We may generalize Mirzakhani's recursion formula to include higher degree κ classes.

Theorem 3.3. *There exist (uniquely determined) rational numbers $\alpha_{\mathbf{L}}$ depending only on $\mathbf{L} \in N^\infty$, such that for any $\mathbf{b} \in N^\infty$ and $d_j \geq 0$, the following recursion relation of mixed ψ and κ intersection numbers holds.*

$$\begin{aligned}
& (2d_1 + 1)!! \langle \kappa(\mathbf{b}) \prod_{j=1}^n \tau_{d_j} \rangle_g \\
&= \sum_{j=2}^n \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}| + d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{L}') \tau_{|\mathbf{L}| + d_1 + d_j - 1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
&+ \frac{1}{2} \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \sum_{r+s=|\mathbf{L}| + d_1 - 2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r + 1)!! (2s + 1)!! \langle \kappa(\mathbf{L}') \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
&+ \frac{1}{2} \sum_{\substack{\mathbf{L} + \mathbf{e} + \mathbf{f} = \mathbf{b} \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=|\mathbf{L}| + d_1 - 2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} (2r + 1)!! (2s + 1)!! \\
&\quad \times \langle \kappa(\mathbf{e}) \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.
\end{aligned}$$

These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \frac{(-1)^{|\mathbf{L}|} \alpha_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! (2|\mathbf{L}'| + 1)!!} = 0, \quad \mathbf{b} \neq 0,$$

namely

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \mathbf{L}' \neq 0}} \frac{(-1)^{|\mathbf{L}'| - 1} \alpha_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! (2|\mathbf{L}'| + 1)!!}, \quad \mathbf{b} \neq 0,$$

with the initial value $\alpha_0 = 1$.

Theorem 3.4.

$$\begin{aligned}
& \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d_1 + 2|\mathbf{L}| + 1)!!}{(2|\mathbf{L}| + 1)!!} \langle \kappa(\mathbf{L}') \tau_{d_1 + |\mathbf{L}|} \prod_{j=2}^n \tau_{d_j} \rangle_g \\
&= \sum_{j=2}^n \frac{(2(d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{b}) \tau_{d_1 + d_j - 1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
&+ \frac{1}{2} \sum_{r+s=|d_1| - 2} (2r + 1)!! (2s + 1)!! \langle \kappa(\mathbf{b}) \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
&+ \frac{1}{2} \sum_{\substack{\mathbf{e} + \mathbf{f} = \mathbf{b} \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=d_1 - 2} \binom{\mathbf{b}}{\mathbf{e}} (2r + 1)!! (2s + 1)!! \\
&\quad \times \langle \kappa(\mathbf{e}) \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.
\end{aligned}$$

Theorem 3.3 and Theorem 3.4 implies each other through Lemma 3.2.

Both Theorems 3.3 and 3.4 are effective recursion formulae for computing higher Weil-Petersson volumes with the three initial values

$$\langle \tau_0 \kappa_1 \rangle_1 = \frac{1}{24}, \quad \langle \tau_0^3 \rangle_0 = 1, \quad \langle \tau_1 \rangle_1 = \frac{1}{24}.$$

From the following Proposition 3.4, we have

$$\langle \kappa(\mathbf{b}) \rangle_g = \frac{1}{2g-2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \kappa(\mathbf{L}') \rangle_g.$$

We have computed a table of $\alpha_{\mathbf{L}}$ for all $|\mathbf{L}| \leq 15$ and have written a Maple program [26] implementing Theorems 3.3 and 3.4.

In fact, we find that ψ and κ classes are compatible in the sense that recursions of pure ψ classes can be neatly generalized to recursions including both ψ and κ classes by the same proof as Proposition 2.4. In view of Theorem 3.8 below, this can be rephrased as differential equations governing generating functions of ψ classes also govern generating functions of mixed ψ and κ classes.

We present some examples below.

Proposition 3.5. *Let $\mathbf{b} \in N^\infty$ and $d_j \geq 0$. Then*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = (2g-2+n) \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g.$$

The above proposition is a generalization of the dilaton equation. In the special case $\mathbf{b} = (m, 0, 0, \dots)$, it has been proved by Norman Do and Norbury [4].

Proposition 3.6. *Let $\mathbf{b} \in N^\infty$. Then*

$$\begin{aligned} \langle \tau_0 \tau_1 \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g &= \frac{1}{12} \langle \tau_0^4 \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g \\ &\quad + \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \underline{n}=I \amalg J}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_0^2 \prod_{i \in I} \tau_{d_i} \kappa(\mathbf{L}) \rangle_{g'} \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \kappa(\mathbf{L}') \rangle_{g-g'}. \end{aligned}$$

The above proposition, together with the projection formula, can be used to derive an effective recursion formula for higher Weil-Petersson volumes [16] (without ψ classes).

Let $\mathbf{s} := (s_1, s_2, \dots)$ and $\mathbf{t} := (t_0, t_1, t_2, \dots)$, we introduce the following generating function

$$G(\mathbf{s}, \mathbf{t}) := \sum_g \sum_{\mathbf{m}, \mathbf{n}} \langle \kappa_1^{m_1} \kappa_2^{m_2} \dots \tau_0^{n_0} \tau_1^{n_1} \dots \rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i \geq 1} s_i^{m_i}$.

Following Mulase and Safnuk [20], we introduce the following family of differential operators for $k \geq -1$,

$$\begin{aligned} V_k = & -\frac{1}{2} \sum_{\mathbf{L}} (2(|\mathbf{L}| + k) + 3)!! \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}!(2|\mathbf{L}| + 1)!!} \mathbf{s}^{\mathbf{L}} \frac{\partial}{\partial t_{|\mathbf{L}|+k+1}} \\ & + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}} + \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} \\ & + \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{48}. \end{aligned}$$

Theorem 3.7. [16, 20] *The recursion of Theorem 3.4 implies*

$$V_k \exp(G) = 0.$$

Moreover, we can check directly that the operators V_k , $k \geq -1$ satisfy the Virasoro relations

$$[V_n, V_m] = (n-m)V_{n+m}.$$

The Witten-Kontsevich theorem states that the generating function for ψ class intersections

$$F(t_0, t_1, \dots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ -function for the KdV hierarchy.

Theorem 3.8. [16, 20] *We have*

$$G(\mathbf{s}, t_0, t_1, \dots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \dots),$$

where p_k are polynomials in \mathbf{s} given by

$$p_k = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{|\mathbf{L}|-1}}{\mathbf{L}!} \mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of \mathbf{s} , $G(\mathbf{s}, \mathbf{t})$ is a τ -function for the KdV hierarchy.

At a final remark, it would be interesting to prove that $\alpha_{\mathbf{L}}$ in Theorem 3.3 are positive for all $\mathbf{L} \in N^{\infty}$. This problem is kindly pointed out to us by a referee.

More generally the question can be formulated as following: two sequences $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ with $\alpha_{\mathbf{0}} = \beta_{\mathbf{0}} = 1$ are said to be inverse to each other if they satisfy

$$\left(\sum_{\mathbf{L}} \alpha_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \right) \cdot \left(\sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \right) = 1.$$

Find sufficient conditions on $\beta_{\mathbf{L}}$ such that $\alpha_{\mathbf{L}} > 0$ for all \mathbf{L} .

We conjecture that $\alpha_{\mathbf{L}}$ are positive when $\sum_{\mathbf{L}} \beta_{\mathbf{L}} s^{\mathbf{L}}$ equals any of the following.

$$\sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!} s^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|-1)!!} s^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!|\mathbf{L}|!} s^{\mathbf{L}}.$$

The latter two arise when we consider Hodge integrals involving λ classes [16].

For works on the positivity criteria of coefficients of reciprocal power series of a single variable, see for example [14]. However it seems there is no literature dealing with the coefficients of reciprocal series of several variables.

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